

Parabolic - hyperbolic boundary layer

MONICA DE ANGELIS *

Abstract

A boundary value problem related to a parabolic higher order operator with a small parameter ε is analyzed. For ε tends to zero, the reduced operator is hyperbolic. When $t \rightarrow \infty$ and $\varepsilon \rightarrow 0$ a parabolic hyperbolic boundary layer appears. In this paper a rigorous asymptotic approximation uniformly valid for all t is established.

1 Introduction

The parabolic operator

$$(1.1) \quad \mathcal{L}_\varepsilon = \partial_{xx}(\varepsilon \partial_t + c^2) - \partial_{tt}$$

is related to the well known Kelvin -Voigt viscoelastic model. Further, it characterizes also the principal part of numerous models with non linear dissipation, such as

$$(1.2) \quad \mathcal{L}_\varepsilon = \beta(u, u_x, u_t).$$

Typical example is the perturbed Sine Gordon equation.[6] Moreover, by

*Facoltà di Ingegneria, Dipartimento di Matematica e Applicazioni, via Claudio 21, 80125, Napoli.

means

of (1.2), wave equations with non linear terms are regularized obtaining a priori estimates and considering the ε parameter vanishing.[1]. Further third operators are also considered to value the Cauchy problem for a second order hyperbolic equation [2] or to regularize parabolic forward- backward equations.[3]

Singular perturbation problem related to equations like (1.2) have interest also to evaluate the influence of the dissipative causes on the wave propagation. [4]. In particular, in the linear case $\beta = f(x, t)$, it's interesting to compare the effects of the diffusion with the pure waves which occur when $\varepsilon = 0$. In this case one has a parabolic- hyperbolic boundary layer with the unique singularity for $t \rightarrow \infty$.

In this paper, we consider the strip problem for equation (1.2) and analyze the singular perturbation problem when $\beta = f(x, t)$ is linear. The Green function related to this problem has been already determined in term of a rapidly decreasing Fourier series.[5].

An appropriate analysis of this series when $\varepsilon \rightarrow 0$ allows to obtain a rigorous asymptotic estimate of the solution, uniformly valid even $t \rightarrow \infty$.

2 Statement of the problem

If $v(x, t)$ is a function defined in

$$\Omega = \{(x, t) : 0 < x < l, \quad t \geq 0\},$$

with l arbitrary positive constant, let IBC the following system of initial-boundary conditions:

$$(2.1) \quad \begin{cases} v(x, 0) = f_0(x), \quad v_t(x, 0) = f_1(x), & x \in [0, l], \\ v(0, t) = \psi_0, \quad v(l, t) = \psi_1, & t \geq 0, \end{cases}$$

with f_i, ψ_i ($i = 0, 1$) regular data.

Consider the operators:

$$(2.2) \quad \mathcal{L}_0 = c^2 \partial_{xx} - \partial_{tt}; \quad \mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon \partial_{xxt}$$

and denote by u_0 and u_ε the solutions of the problems:

$$Problem P_0 : \quad cal L_0 u_0 = -f \text{ with IBC } (2.1)$$

$$Problem P_\varepsilon : \quad cal L_\varepsilon u_\varepsilon = -f \text{ with IBC } (2.1),$$

where $f(x, t)$ is a prefixed source term.

To obtain a rigorous approximation of u_ε when $\varepsilon \rightarrow 0$, we put

$$(2.3) \quad u(x, t, \varepsilon) = u_0(x, t) + \varepsilon r(x, t, \varepsilon)$$

where u_0 is the well-known solution of the classical problem P_0 , while the error term represent the solution of the $Problem P_r$:

$$(2.4) \quad \begin{cases} cal L \varepsilon r = -F(x, t) & (x, t) \in \Omega \\ v(x, 0) = f_0(x), \quad v_t(x, 0) = f_1(x), & x \in [0, l], \\ v(0, t) = \psi_0, \quad v(l, t) = \psi_1, & t \geq 0, \end{cases}$$

with $F(x, t) = \partial_x x t u_0$. Therefore, following results in [1], one has:

$$(2.5) \quad r(x, t, \varepsilon) = - \int_0^l d\xi \int_0^t F(\xi, \tau, \varepsilon) G(x, \xi, t - \tau) d\tau$$

where $G(x, \xi, t)$ is the Green function related to \mathcal{L}_ε operator.

In particular, for all integer $n \geq 1$, letting:

$$(2.6) \quad \gamma_n = \frac{\pi}{l} n \quad a_n = \frac{\varepsilon}{2} \gamma_n^2 \quad k = \frac{2cl}{\pi \varepsilon}$$

$$b_n = \gamma_n c \sqrt{1 - (n/k)^2} \quad H_n = e^{/a_n} \frac{sen(b_n t)}{b_n},$$

i

$$(2.7) \quad G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} H_n(t) \sin \gamma_n x \sin \gamma_n \xi$$

with

$$(2.8) \quad H_n(t) = \frac{e^{-bn^2t}}{bn^2\sqrt{1-(k/n)^2}} \sinh(bn^2t\sqrt{1-(k/n)^2})$$

and

$$(2.9) \quad b = \frac{\pi^2}{2l^2}\varepsilon = q\varepsilon, \quad k = \frac{2cl}{\pi\varepsilon} \quad \gamma_n = \frac{\pi}{l}n.$$

Now, denote with $u(x, t)$ the solution of the reduced problem obtained by (2.1) with $\varepsilon = 0$. To obtain an asymptotic approximation for $w(x, t)$ when $\varepsilon \rightarrow 0$, we put:

$$(2.10) \quad w(x, t, \varepsilon) = e^{-\varepsilon t}u(x, t) + r(x, t, \varepsilon)$$

where the error $r(x, t, \varepsilon)$ must be evaluated.

By means of standard computations one verifies that $r(x, t, \varepsilon)$ is the solution of the problem:

consider the operators

$$(2.11) \quad \begin{cases} \partial_{xx}(\varepsilon r_t + c^2 r) - \partial_{tt}r = f(x, t, \varepsilon), & (x, t) \in D, \\ r(x, 0) = 0, \quad r_t(x, 0) = 0, & x \in [0, l], \\ r(0, t) = 0, \quad r(l, t) = 0, & 0 < t < T, \end{cases}$$

where the source term $f(x, t, \varepsilon)$ is:

$$(2.12) \quad f(x, t, \varepsilon) = F(x, t)(1 - e^{-\varepsilon t}) + e^{-\varepsilon t}[-\varepsilon\lambda_t + \varepsilon^2(u + u_{xx})]$$

with $\lambda = 2u + u_{xx}$.

The problem (2.11) has already been solved in [5] and the solution is given by:

$$(2.13) \quad r(x, t, \varepsilon) = - \int_0^l d\xi \int_0^t f(\xi, \tau, \varepsilon) G(x, \xi, t - \tau) d\tau$$

where $G(x, \xi, t)$ is:

$$(2.14) \quad G(x, \xi, t) = \frac{2}{l} \sum_{n=1}^{\infty} H_n(t) \sin \gamma_n x \sin \gamma_n \xi$$

with

$$(2.15) \quad H_n(t) = \frac{e^{-bn^2t}}{bn^2\sqrt{1-(k/n)^2}} \sinh(bn^2t\sqrt{1-(k/n)^2})$$

and

$$(2.16) \quad b = \frac{\pi^2}{2l^2}\varepsilon = q\varepsilon, \quad k = \frac{2cl}{\pi\varepsilon} \quad \gamma_n = \frac{\pi}{l}n.$$

3 Analysis of $G(x, t, \xi, \varepsilon)$ when ε tends to zero.

In order to investigate the behaviour of the Green function G when parameter $\varepsilon \rightarrow 0$, referring to the function G defined in (2.14), let:

$$(3.1) \quad H_n^1(t) = \frac{e^{-bn^2t}}{bn^2\sqrt{(k/n)^2-1}} \sin bn^2t\sqrt{(k/n)^2-1}$$

and

$$(3.2) \quad G(x, \xi, t) = \frac{2}{l} \left\{ \sum_{n=1}^{[k]} H_n^1(t) + \sum_{[k]+1}^{\infty} H_n(t) \right\} \sin \gamma_n x \sin \gamma_n \xi = G_1 + G_2.$$

If α is an arbitrary constant such that:

$$(3.3) \quad 1/2 < \alpha < 1, \quad \bar{n} = \frac{2cl}{\pi\varepsilon^\alpha},$$

the term G_1 of G can be given the forms:

$$(3.4) \quad G_1(x, \xi, t) = \frac{2}{l} \left\{ \sum_{n=1}^{[\bar{n}]} H_n^1(t) + \sum_{[\bar{n}]+1}^{[k]} H_n^1(t) \right\} \sin \gamma_n x \sin \gamma_n \xi .$$

It is easy to prove that if $1 \leq n \leq [\bar{n}]$ it holds:

$$(3.5) \quad \sqrt{(k/n)^2 - 1} \geq \frac{\sqrt{1 - \varepsilon^{2(1-\alpha)}}}{\varepsilon^{1-\alpha}}; \quad e^{-bn^2 t} \leq e^{-qt\varepsilon}.$$

Otherwise, if $[\bar{n}] + 1 \leq n \leq [k]$:

$$(3.6) \quad \sqrt{(k/n)^2 - 1} \geq \frac{\sqrt{\pi\varepsilon\beta}\sqrt{4cl - \beta\pi\varepsilon}}{(2cl - \pi\varepsilon\beta)}; \quad e^{-bn^2 t} \leq e^{-2c^2 t/\varepsilon^{2\alpha-1}},$$

where $0 < \beta < 1$. In particular, if k is an integer we will assume $\beta = 1$ and we will explicitly consider the term with $n = k$, having $te^{-2c^2 t/\varepsilon}$.

Since (3.5) and (3.6), the following inequality holds:

$$(3.7) \quad |G_1(x, \xi, t)| \leq N(\varepsilon)\varepsilon^{-\alpha}e^{-qt\varepsilon} + N_1(\varepsilon)\varepsilon^{-3/2}e^{-c^2 t/\varepsilon^{2\alpha-1}},$$

where

$$(3.8) \quad N(\varepsilon) = \frac{2\zeta(2)}{ql}[1 - \varepsilon^{2(1-\alpha)}]^{-1/2}; \quad N_1(\varepsilon) = \frac{2\zeta(2)(2cl - \pi\varepsilon\beta)}{ql\sqrt{\pi\beta}\sqrt{4cl - \beta\pi\varepsilon}}$$

and $\zeta(2)$ is the Riemann zeta function.

There remains to determine an upper bound for hyperbolic terms. This may be done using inequalities proved in [5]. So, being $\forall n \geq [\bar{k}] + 1$:

$$(3.9) \quad bn^2 t(1 \pm \sqrt{1 - (k/n)^2}) \geq c^2/\varepsilon,$$

and since

$$(3.10) \quad \sqrt{1 - (k/n)^2} \geq \frac{\pi\varepsilon(1 - \beta)[4cl + \pi\varepsilon(1 - \beta)]}{2cl + \pi\varepsilon(1 - \beta)},$$

with $\beta \equiv 0$ if k is an integer, we can write:

$$(3.11) \quad |G_2(x, \xi, t)| \leq C_1(\varepsilon) \varepsilon^{-2} e^{-c^2 t/\varepsilon}$$

where

$$(3.12) \quad C_1(\varepsilon) = \frac{2\zeta(2)[cl + \pi\varepsilon(1 - \beta)]}{ql\pi(1 - \beta)[4cl + \pi\varepsilon(1 - \beta)]}.$$

The previous results lead to prove the following

Theorem 3.1 - *The Green function $G(x, \xi, t)$ defined in (2.14) converges absolutely for all $(x, t) \in D$. Moreover, indicating by $M(\varepsilon) = \max\{N_1(\varepsilon) \varepsilon^{-3/2}, C_1(\varepsilon) \varepsilon^{-2}\}$, it holds:*

$$(3.13) \quad |G(x, \xi, t)| \leq N(\varepsilon)\varepsilon^{-\alpha}e^{-qt\varepsilon} + M(\varepsilon)e^{-c^2 t/\varepsilon^{2\alpha-1}}.$$

4 Asymptotic approximation

Now, we are able to estimate function $r(x, t, \varepsilon)$ i.e. it is possible to have an upper bound for the solution of problem (2.11).

In fact, recalling expression (2.12)-(2.13), it holds:

$$(4.1) \quad |r(x, t, \varepsilon)| \leq l\varepsilon \int_0^t e^{-\varepsilon\tau} \{|\lambda_t(x, \tau)| + \varepsilon|\lambda - u|\} |G(x, \xi, t - \tau)| d\tau + \\ + l \int_0^t |F(x, \tau)| |1 - e^{-\varepsilon\tau}| |G(x, \xi, t - \tau)| d\tau.$$

So, choosing:

$$(4.2) \quad 3/4 < \alpha < 1 \quad \text{and} \quad 2(2\alpha - 1)^{-1} < \delta < 1,$$

let:

$$(4.3) \quad \beta = \delta(2\alpha - 1) - 1/2, \quad 0 < \gamma < 1.$$

So, if

$$(4.4) \quad \eta = \min\{\beta, \gamma, 1 - \alpha, 1/2\};$$

and

$$(4.5) \quad A = \max\{\sup_D |F|, \sup_D |\lambda - u|, \sup_D |\lambda_t|\}$$

the following lemma holds:

Lemma 4.1 - *If the function $f(x, t, \varepsilon)$ defined in (2.12) is a continuous function in D with continuous derivative with respect to x , then the function $r(x, t, \varepsilon)$ satisfies the inequality :*

$$(4.6) \quad |r| \leq A\varepsilon^\eta \{t^2 Z(\varepsilon) + tY(\varepsilon) + \{t^{2-\delta} + t^{1-\delta}\}W(\varepsilon) + t^{1-\gamma}V(\varepsilon)\} + \\ + A\{U(\varepsilon) e^{-c^2 t/\varepsilon} + S(\varepsilon)\}$$

with

$$(4.7) \quad Z(\varepsilon) = N(\varepsilon)/2; \quad Y(\varepsilon) = \max\{2N(\varepsilon), N_1(\varepsilon)\}$$

$$W(\varepsilon) = N_1(\varepsilon)(\delta/e)^\delta \quad V(\varepsilon) = C(\varepsilon)[(1 + \gamma)/e]^{1+\gamma}(1 - \gamma)^{-1}$$

$$U(\varepsilon) = 2q\varepsilon/c^2\zeta(2) + C(\varepsilon)/c^2 + \varepsilon/c^2; \quad S(\varepsilon) = 2q\varepsilon\zeta(2)/c^2 + \varepsilon/c^2.$$

Proof- Since the well known inequality [7]:

$$(4.8) \quad e^{-x} \leq [a/(ex)]^a \quad \forall a > 0, \forall x > 0$$

and (4.1), it holds:

$$(4.9) \quad |r| \leq Al[\varepsilon^{1-\alpha}(t^2/2 + t + \varepsilon t)N + N_1[\varepsilon^\beta(\delta/e)^\delta(t^{2-\delta} + t^{1-\delta}) + t\sqrt{\varepsilon}] +$$

$$+ C_1(\varepsilon)[(1 + \gamma)/e^{1+\gamma}(1 - \gamma)^{-1}t^{1-\gamma}\varepsilon^\gamma + \varepsilon/c^2 + e^{-c^2t/\varepsilon}(\varepsilon/c^2 + c^{-2}) +$$

$$+ 2\varepsilon\zeta(2)q/c^2e^{-c^2t/\varepsilon} + 2q\varepsilon/c^2\zeta(2),$$

from which, taking into account (4.2), (4.3), lemma follows.

In this way , if we consider the set

$$(4.10) \quad Q_\varepsilon = \{(x, t) : 0 \leq x \leq l, 0 < t < \varepsilon^{-\eta/2}\}$$

the following theorem holds:

Theorem 4.1 - *When $\varepsilon \rightarrow 0$, the solution of the parabolic problem (2.1) verifies the following asymptotic estimate*

$$(4.11) \quad w(x, t, \varepsilon) = e^{-\varepsilon t}u(x, t) + r(x, t, \varepsilon)$$

where the error $r(x, t, \varepsilon)$ is uniformly bounded every where in Q_ε .

References

- [1] A.I. Kozhanov N. A. Lar'kin, *Wave equation with nonlinear dissipation in noncylindrical Domains*, Dokl. Math 62, 2, 2000 17-19
- [2] V.P. Maslov, P. P. Mosolov *Non linear wave equations perturbed by viscous terms* Walter deGruyter Berlin N. Y. 2000 pp 329
- [3] G. I. Barenblatt, M. Bertsch, R. Del Passo M. Ughi, *it Adegenerate pseudoparabolic regularization of a nonlinear forward- backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow*. Siam J. Math Anal 24, no 6 1414-1439 (1993).

- [4] Ali Nayfeh *A comparison of perturbation methods for nonlinear hyperbolic waves* in Proc. Adv. sem. Wisconsin no 45 (1980).
- [5] M. De Angelis *Asymptotic analysis for the strip problem related to a parabolic third- order operator*, Appl.Math.Lett 14,4 pp 425-430 (2001)
- [6] B. D'Acunto, M. De Angelis, P. Renno, *Fundamental solution of a dissipative operator*, Rend. Acc. Sc. Fis. Mat. (1997)
- [7] D.S. Mitrinovic *Analytic Inequalities* Springer 1970